

The Ruijsenaars-Schneider Model

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Abstract

We seek to clarify some of the physical aspects of the Ruijsenaars-Schneider models. This important class of models was presented as a relativistic generalisation of the Calogero-Moser models but, as we shall argue, this description is misleading. It is far better to simply view the models as a one-parameter generalisation of Calogero-Moser models. By viewing the models as describing certain eigenvalue motions we can appreciate the generic nature of the models.

1 Introduction

The Calogero-Moser models [1] are completely-integrable, Hamiltonian systems describing (non-relativistic) particle dynamics with pairwise interaction potentials of the form $1/x^2$, $1/\sin^2 x$, $1/\sinh^2 x$ (and in general the Weierstrass \wp functions). The models are rather generic, which accounts for their importance in various branches of theoretical physics from solid state physics to particle physics [2, 3, 4]: they appear when describing the eigenvalue motion of certain matrices [5]; the pole motions of the solitons of various PDE's are described by the model (with possible constraints) [6]; the quantum mechanics [7] of these models has been connected with the transmission properties of wires [8] and Conformal Field Theory [9]. A rich algebraic structure is being uncovered behind the models [10].

The successes of the Calogero-Moser systems have naturally led to an expectation that their “relativistic” versions, *if any*, might play similar roles in connection with integrable relativistic quantum field theories. Examples of integrable relativistic quantum field theories include the sine-Gordon model and affine Toda field theories (the latter being constructed from the various affine Lie algebras). Thanks to the infinite number of conserved quantities which characterises the integrability of quantum field theories, no particle creation and annihilation are allowed in such theories and their N -particle S -matrices are factorised into a product of $N(N-1)/2$ two-particle S -matrices. The expectation that an integrable relativistic field theory might equivalently and simply be described in terms of some integrable “relativistic” particle dynamics was speculated by Ruijsenaars in [11] and appears more explicitly in [12], where Ruijsenaars and Schneider describe the motivation lying behind the discovery of their model. Here the model was proposed as a “relativistic” (or one-parameter c , the velocity of light) generalisation of Calogero-Moser model. (The model is variously referred to as the “relativistic” Calogero-Moser model or Ruijsenaars-Schneider model. For reasons we later give, we prefer the latter nomenclature.)

Our aim in the following note is to further explicate these models and in particular the role of “relativistic invariance”. The viewpoint described below is that the Ruijsenaars-Schneider system is an important and rather generic integrable system, but to describe it as expressing “relativistic particle dynamics” is quite misleading. The importance of the Ruijsenaars-Schneider system cannot be underestimated: it arises as a particular form of eigenvalue motion in much the same way as the Calogero-Moser model does, and this eigenvalue motion is relevant in many physical settings. Just as the Calogero-Moser model is related to particular solutions of PDE's, the Ruijsenaars-Schneider model is also connected with particular soliton solutions of for example the KdV, mKdV and sine-Gordon equations.

It has been connected with the gauged WZW model [13]. A rich algebraic structure is also being uncovered [14] for the model and spin-generalisations [15] of the Ruijsenaars-Schneider model are known, paralleling¹ the spin-generalisations [16] of the Calogero-Moser model. The solitons of the a_n affine Toda field theories with imaginary coupling constant have been related [17] to these spin-generalisations extending the sine-Gordon soliton and Ruijsenaars-Schneider correspondence mentioned above [18]. But clearly if the same model is related to the relativistically invariant sine-Gordon equation and also the relativistically noninvariant KdV equation (and others), the simple notion of “relativistic particle dynamics” needs clarification.

The first difficulty one usually encounters when seeking to describe “relativistic particle dynamics” is how any theory with a single time can be compatible with causality. Any interaction Hamiltonian or Lagrangian depending on the coordinates and momenta of the other particles in a single time formulation is by definition ‘action-at-a-distance’. The time evolution of the positions and momenta is determined by the positions and momenta of the other particle *at the same time*. In order for this to happen each particle must be able to ‘know’ the coordinates and momenta of the other particles *instantly*. This obviously breaks Einstein’s causality. One possible way to circumvent the above difficulty is of course to adopt an interaction potential of zero range, namely the delta function potential. In two and higher space dimensions the delta function potential is too singular to be treated properly [22], but as is well known in quantum mechanics the delta function potential in one space dimension can easily be handled. In fact in this case relativistic many particle theory can be properly formulated [23] and the particle coordinates and times obey the Lorentz transformation and together with the generators of space and time translations and boost satisfy the Poincaré algebra. However, with any long range interaction $f(q)$ and a single time formalism the incompatibility of ‘action-at-a-distance’ with Einstein’s causality remains. Actually the Ruijsenaars-Schneider models have several “times” corresponding to different commuting flows H_j ,

$$(q(t_1, t_2, \dots t_l), \theta(t_1, t_2, \dots t_l)) = \exp \left(\sum_{j=1}^l t_j H_j \right) (q(0), \theta(0)), \quad (1.1)$$

and the solutions of the PDE’s mentioned above require the evolution to be determined with respect to each of these times. In particular, when the flows H_1 and H_{-1} are both present and so $q_j = q_j(t, x)$, the theory exhibits a Poincaré invariance, but as we shall argue the theory is not relativistically invariant in the sense suggested by the “non-relativistic” limit

¹In this context we note that a Hamiltonian formulation for these spin-generalisations is still lacking.

given by Ruijsenaars and Schneider. Indeed the presence of two “times” or flows means we are not dealing with a traditional notion of relativistic dynamics and the standard “no-go” theorems [24] are correspondingly avoided. Because the coordinates $q(t, x)$ and $\theta(t, x)$ of the Poincaré invariant Ruijsenaars-Schneider model are parameterized by Minkowski space it may be thought that what we have here is some, albeit unusual, field theory. We shall show however that the solutions $q(t, x)$, $\theta(t, x)$ of the Ruijsenaars-Schneider model do not describe the dynamical time-evolution typical of field theory and are more akin to those of a topological field theory in the sense that they do not possess dynamical degrees of freedom.

The Note is organised as follows. In section two some salient features of the Ruijsenaars-Schneider model are briefly reviewed to set the stage and notation. We view the Ruijsenaars-Schneider model as describing the motion of eigenvalues of matrices of certain type, a simple generalisation of the Calogero-Moser situation. Then the connection with the N -soliton solutions of various soliton equations (KdV and sine-Gordon, etc) is briefly mentioned. In section three the nature of the “relativistic invariance” of the Ruijsenaars-Schneider model is clarified starting with its “non-relativistic” limit. The many “times” formulation and the Poincaré invariance of the theory is also discussed. Section four discusses the field theory aspects of the Ruijsenaars-Schneider model. In section five we dwell upon the possible connection between integrable quantum field theories with exact factorisable S-matrices and the Ruijsenaars-Schneider model. The uncertainty principle of quantum theory plays an important role here. The final section is for summary and discussion. Throughout we will try to use the notation of Ruijsenaars and Schneider [12] or Ruijsenaars [19] as far as possible.

2 The Ruijsenaars-Schneider Model

In this section we first review the salient features of the Ruijsenaars-Schneider model to fix the notation; the details may be found in [12, 19]. Having done this we next review how the model arises when describing the eigenvalue motion of a particular (possibly partial) differential matrix equation. This is our perspective on the models, and others may differ here. Theorems pertaining to these eigenvalue motions may be found in [20]. We conclude with the connection between this model and soliton equations.

2.1 Salient Features

The dynamical variables of the Ruijsenaars-Schneider theory are the “rapidity” θ_j and its canonically conjugate “position” q_j , satisfying the following Poisson bracket relations:

$$\{q_j, q_k\} = \{\theta_j, \theta_k\} = 0, \quad \{q_j, \theta_k\} = \delta_{jk}, \quad j, k = 1, \dots, N. \quad (2.1)$$

We see from (2.1) that if the “rapidity” θ_j is taken to be dimensionless, then q_j has the dimensions of action; the product of any two canonical variables has the dimensions of action. The Hamiltonian H , the “space-translation” generator P and “boost” generator B are given by

$$H = mc^2 \sum_{j=1}^N \cosh \theta_j \prod_{k \neq j} f\left(\frac{q_j - q_k}{A}\right), \quad (2.2)$$

$$P = mc \sum_{j=1}^N \sinh \theta_j \prod_{k \neq j} f\left(\frac{q_j - q_k}{A}\right), \quad (2.3)$$

$$B = -\frac{1}{c} \sum_{j=1}^N q_j, \quad (2.4)$$

where c is the velocity of light and A is a constant having the dimension of the action (see section three for more detail). They satisfy the following relations

$$\{H, P\} = 0, \quad \{H, B\} = P, \quad \{P, B\} = H/c^2, \quad (2.5)$$

provided $f^2(z)$ equals $\lambda + \mu\wp(z)$, including its trigonometric, hyperbolic and rational degenerate cases. These are the relations that the generators of the two-dimensional Poincaré algebra should satisfy. It is an added bonus that this choice of the function f also ensures the existence of N independent, Poisson commuting conserved quantities, and so the Ruijsenaars-Schneider model is completely integrable. Typical of the conserved quantities constructed are $H_{\pm 1}$ where

$$H_{\pm 1} = mc^2 \sum_{j=1}^N e^{\pm \theta_j} \prod_{k \neq j} f\left(\frac{q_j - q_k}{A}\right), \quad (2.6)$$

and so $H = (H_1 + H_{-1})/2$ and $P = (H_1 - H_{-1})/2$ in the above.

Contrary to [12, 19] we have emphasised the appearance of the dimensionful parameter A necessary² to define the theory. The Lagrangians associated with these systems are rather unusual and have some interesting features. The ‘Lagrangian’ associated with (say) H_+ is

$$\mathcal{L} = \sum_{j=1}^N \dot{q}_j \left(\ln \frac{\dot{q}_j}{mc^2} - 1 - \ln \prod_{k \neq j} f\left(\frac{q_j - q_k}{A}\right) \right), \quad (2.7)$$

² In [19] Ruijsenaars chooses to work with the variables $\bar{q}_j = mc q_j$ and $\bar{\theta}_j = \theta_j/mc$. In this case a dimensionful length scale $A/mc = L$ must appear in the functions (2.22) of that reference.

and we remark that the first term on the right here behaves as an ‘entropy’. For the remainder of this section we will set $A = m = c = 1$, but will reinstate these constants at later junctures in our discussion.

2.2 Eigenvalue Motion and the Ruijsenaars-Schneider model

The Ruijsenaars-Schneider theory and its generalisations may be viewed as describing the motion of the eigenvalues of matrices of certain type. For example, let V be a real, symmetric, positive-definite $N \times N$ matrix whose ‘time’ dependence satisfies

$$\partial V = \Lambda V + V \Lambda, \quad (2.8)$$

where Λ is a constant matrix. As we shall now review, the eigenvalue motion corresponding to (2.8) leads to a mechanical system that is directly analogous to the linear motions associated with the Calogero-Moser model. Here the constancy of Λ plays the same role as the constants of motion in the Calogero-Moser situation. The Ruijsenaars-Schneider model arises when ∂V is further assumed to be of a specific form; this restriction is directly analogous to the constraint on the angular momentum made for the Calogero-Moser model. We will later give examples of such $N \times N$ matrices satisfying (2.8) that are to be found in connection with the N -soliton solutions of some soliton theories.

Let V be diagonalised by the orthogonal matrix U and set

$$Q = UVU^{-1} = \text{diag}(\exp(q_1), \dots, \exp(q_N)), \quad M = \partial U U^{-1},$$

where M is an anti-symmetric matrix $M = -M^t$. Then upon setting $L = U \Lambda U^{-1}$ we obtain the Lax equation

$$\partial L = [M, L], \quad \partial Q = [M, Q] + U \partial V U^{-1} = [M, Q] + L Q + Q L. \quad (2.9)$$

From this it is easy to obtain

$$L_{jj} = (1/2) \partial q_j \quad (2.10)$$

and (for $j \neq k$)

$$M_{jk} = \left(\frac{Q_j + Q_k}{Q_j - Q_k} \right) L_{jk} = \coth((q_j - q_k)/2) L_{jk}.$$

Substituting these into the Lax equation produces (with $\dot{q}_j = \partial q_j$) the equations of motion:

$$\dot{L}_{jj} = \frac{1}{2} \ddot{q}_j = 2 \sum_{k \neq j} \coth((q_j - q_k)/2) L_{jk} L_{kj}, \quad (2.11)$$

$$\begin{aligned} \dot{L}_{jk} = & \frac{1}{2} \coth((q_j - q_k)/2) (\dot{q}_k - \dot{q}_j) L_{jk} \\ & + \sum_{l \neq j, k} (\coth((q_j - q_l)/2) - \coth((q_l - q_k)/2)) L_{jl} L_{lk}, \quad (j \neq k). \end{aligned} \quad (2.12)$$

As shown in [17] these are the spin-generalised Ruijsenaars-Schneider equations [15] with certain constraints.

The (non-spin) model of Ruijsenaars-Schneider now results when \dot{V} may be expressed as

$$\dot{V}_{jk} = e_j e_k, \quad j, k = 1, \dots, N, \quad (2.13)$$

for some real vector e (e_j being its j -th component). Then with $\tilde{e} = Ue$ we find

$$L_{jk} = \frac{\tilde{e}_j \tilde{e}_k}{\exp(q_j) + \exp(q_k)}.$$

Since we know the diagonal elements of L explicitly in terms of the q_j we have

$$L_{jk} = \frac{\sqrt{\dot{q}_j \dot{q}_k}}{\cosh((q_j - q_k)/2)}. \quad (2.14)$$

This may then be substituted into (2.11) to give

$$\ddot{q}_j = 2 \sum_{k \neq j} \frac{\dot{q}_j \dot{q}_k}{\sinh(q_j - q_k)}. \quad (2.15)$$

These are the equations of motion for (either $H_{\pm 1}$)

$$H_{\pm 1} = \sum_{j=1}^N e^{\pm \theta_j} \prod_{k \neq j}^N \coth\left(\frac{q_j - q_k}{2}\right), \quad (2.16)$$

with conjugate variables q_j, θ_j , satisfying the canonical Poisson bracket relations (2.1). In this case (2.12) is then identically satisfied.

Now $H = (H_1 + H_{-1})/2$ and $P = (H_1 - H_{-1})/2$ are particular cases of (2.2) and (2.3). Thus the hyperbolic Ruijsenaars-Schneider model may be identified with the eigenvalue motion just described. Other (possibly difference [25]) matrix equations correspond to the different functions f appearing in the Ruijsenaars-Schneider model. Further, if L is the Lax matrix associated with the Ruijsenaars-Schneider theory above then each of the flows corresponding to $H_k = (1/k) \text{tr } L^k$ is also conserved and $\{H_k, H_l\} = 0$; these give the conserved quantities associated with the model. Upon setting

$$\mathcal{H}_k = (H_k + H_{-k})/2, \quad \mathcal{P}_k = (H_k - H_{-k})/2, \quad \mathcal{B} = - \sum_j^N q_j, \quad (2.17)$$

we have

$$\{\mathcal{H}_k, \mathcal{P}_k\} = 0, \quad \{\mathcal{H}_k, \mathcal{B}\} = \mathcal{P}_k, \quad \{\mathcal{P}_k, \mathcal{B}\} = \mathcal{H}_k. \quad (2.18)$$

For any k this has the form of the two dimensional Poincaré algebra. Also note from $\sum_j \ddot{q}_j = 0$ that \mathcal{B} evolves linearly with respect to the H_1 flow.

2.3 Connection with N -soliton Solutions

The Ruijsenaars-Schneider theory appears in the study of N -soliton solutions of equations whose tau functions have the form

$$\tau = \sum_{\epsilon} \exp \left(\sum_{j < k} \epsilon_j \epsilon_k B_{jk} + \sum_j \epsilon_j \zeta_j(t, x) \right). \quad (2.19)$$

In the above the ϵ indicates a summation over all possible combinations of ϵ_j taking the values 0 or 1, and the indices j and k take values in $\{1, \dots, N\}$. The expression (2.19) is a rather generic form of the soliton tau function for an integrable PDE, the precise nature of B_{jk} and ζ_j depending on the particular PDE being considered. It may be viewed as a degeneration of the theta function solutions of the PDE given via algebraic geometry in which the ϵ_j 's run over all of the integers.

Now in appropriate circumstances this tau function can be written in terms of determinants. Thus for the Sine-Gordon equation we have

$$e^{i\beta\phi} = \frac{\det(1 - V)}{\det(1 + V)},$$

while for the KdV equation

$$\dot{u} - uu' + u''' = 0$$

we have $u = -2(\ln \tau)''$ where

$$\tau = \det(1 + V).$$

In both cases the matrix has the form

$$V_{jk} = \frac{\sqrt{X_j X_k}}{\mu_j + \mu_k}, \quad (2.20)$$

where

$$X_j = 2 a_j \exp(\xi_j(t, x)) \quad (2.21)$$

and

$$\xi_j(t, x) = \xi_j(0) + \mu_j^3 t - \mu_j x, \quad (KdV), \quad (2.22)$$

$$\xi_j(t, x) = \xi_j(0) + \mu_j^{-1} x_- + \mu_j x_+, \quad (SG). \quad (2.23)$$

For the x flow of the KdV equation and either of the SG flows corresponding to the light cone coordinates x_{\pm} , the matrix equation (2.8) is satisfied and the Ruijsenaars-Schneider theory (2.16) ensues. For the SG equation μ_j is related to a rapidity. For other soliton equations that may be expressed in terms of matrices of the form (2.20) and (2.21) the ‘times’ linear in $\mu_j^{\pm 1}$ yield the Ruijsenaars-Schneider theory (2.2).

3 Relativistic Invariance

We wish now to examine the “relativistic invariance” of the theories presented by Ruijsenaars and Schneider as ‘*a class of finite-dimensional integrable systems that may be viewed as relativistic generalizations of the Calogero-Moser systems.*’ In the first part of this section we argue that the Ruijsenaars-Schneider theory is not relativistically invariant in the natural variables suggested by this description. This is why we believe the description of Ruijsenaars-Schneider models as ‘relativistic Calogero-Moser models’ is misleading. Indeed, the “non-relativistic” limit of these models requires an explicit scaling of the dimensionful coupling constant A needed to define these theories, and it is unclear why this should be described as a “non-relativistic” limit. Rather the relativistic invariance of the theories, and that we feel intended by Ruijsenaars and Schneider, is more subtle. We shall go on in the latter subsection to investigate this, but note that this relativistic invariance does not yield relativistically invariant particle *dynamics*.

At the outset it is instructive to see in what sense the Ruijsenaars-Schneider models yield the corresponding Calogero-Moser models as non-relativistic limits. For the sake of both ease and concreteness consider

$$f^2\left(\frac{q_j - q_k}{A}\right) = 1 + \frac{\alpha^2}{\sinh^2\left(\frac{q_j - q_k}{A}\right)};$$

similar results hold for the other potentials. (Here A is to be identified with $2/\mu$ in (4.12) of [12].) Under the following scalings (which preserve the Poisson bracket relations for the new variables \bar{q}_j and $\bar{\theta}_j$)

$$\theta_j = \frac{\bar{\theta}_j}{c}, \quad q_j = c \bar{q}_j, \quad \alpha = \frac{v}{c}, \quad A = c A' \quad (3.1)$$

we find

$$H_{nr} = \lim_{c \rightarrow \infty} (H - Nmc^2) = \frac{m}{2} \sum_{j=1}^N \bar{\theta}_j^2 + \sum_{i \neq j} \frac{mv^2}{\sinh^2\left(\frac{\bar{q}_j - \bar{q}_k}{A'}\right)}. \quad (3.2)$$

Upon using the identification

$$q_j = x_j mc \cosh \theta_j, \quad p_j = mc \sinh \theta_j, \quad j = 1, \dots, N. \quad (3.3)$$

where now

$$\{x_j, x_k\} = \{p_j, p_k\} = 0, \quad \{x_j, p_k\} = \delta_{jk}, \quad j, k = 1, \dots, N, \quad (3.4)$$

Ruijsenaars and Schneider then express this as

$$H_{nr} = \frac{1}{2} \sum_{j=1}^N \frac{p_j^2}{m} + \sum_{i \neq j} \frac{mv^2}{\sinh^2((x_j - x_k)/L)}, \quad (3.5)$$

which is the Hamiltonian of an appropriate Calogero-Moser model. (Here we write $A' = mL$, L being a constant having the dimension of length. The constant v has the dimension of the velocity.) Similarly they obtain

$$P_{nr} = m \sum_{j=1}^N \bar{\theta}_j = \sum_{j=1}^N p_j, \quad (3.6)$$

$$B_{nr} = -m \sum_{j=1}^N x_j. \quad (3.7)$$

As Ruijsenaars and Schneider remark, this limit has required scaling the coupling constants of the theory. Indeed, however one takes this limit, one cannot avoid³ scaling the dimensionful ‘coupling constant’ A . Certainly this analysis shows that the Ruijsenaars-Schneider models reduce to the Calogero-Moser models in a particular scaling limit, but it is not clear that this should be described physically as a “non-relativistic” limit. Only by (infinitely) shifting the Hamiltonian do the generators of the Poincaré algebra reduce to the Galilei generators and, as we shall now show, the Ruijsenaars-Schneider model is not relativistically invariant in the naive sense one would expect for a theory described as a “relativistic generalisation” of the Calogero-Moser model. It seems altogether better to describe the Ruijsenaars-Schneider theory as a one-parameter extension of the Calogero-Moser models.

Now Einstein’s special relativity simply states that an ‘event’ is a point in *Minkowski space*. The essential point is that special relativity is more than a closed Poincaré algebra (like (2.5)): one also needs the Minkowski space upon which it acts via the inhomogeneous Lorentz (Poincaré) transformation

$$\begin{pmatrix} t'_0 \\ x'_0 \end{pmatrix} = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix} \begin{pmatrix} t_0 \\ x_0 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}. \quad (3.8)$$

For relativistically invariant particle *dynamics* one further needs *dynamical variables* directly related with the *Minkowski* positions and momenta. Now by describing their models as ‘relativistic generalisations’ of the Calogero-Moser system, one is naturally led to expect that the q_j or the x_j , arising in the “non-relativistic limit” above, are possible Minkowski space variables. Indeed if we wish the Hamiltonian (2.2) to be space translation invariant –it is manifestly time-translation invariant since the Hamiltonian H does not contain the time explicitly– we must identify the q_j as the Minkowski space variables since (2.2) depends only on their differences. Let us see that neither q_j or x_j are possible Minkowski space variables. To this end we record the following actions of the “space-translation” generator

³ It may at first appear that the β scaling given in [12] avoids the scaling of the parameter μ , which plays the role of $1/A$ here. This is not really the case, for the q_j variables must also be scaled to preserve the Poisson bracket relations; the three different scalings given in [12, 19] are identical.

P and “boost” generator B on q_j and θ_j :

$$\delta_P q_j = \{q_j, P\} = mc \cosh \theta_j \prod_{k \neq j} f(q_j - q_k), \quad \delta_P \theta_j = \{\theta_j, P\} \neq 0, \quad (3.9)$$

$$\delta_B \theta_j = \{\theta_j, B\} = \frac{1}{c}, \quad \delta_B q_j = \{q_j, B\} = 0. \quad (3.10)$$

These imply

$$\delta_P x_j = \prod_{k \neq j} f(q_j - q_k) - x_j \tanh \theta_j \{\theta_j, P\}, \quad \delta_P p_j \neq 0, \quad (3.11)$$

and that the finite transformations under “boosts” are

$$\theta'_j = \theta_j + \frac{\alpha}{c}, \quad q'_j = q_j, \quad \text{or} \quad x'_j = x_j \frac{\cosh \theta_j}{\cosh(\theta_j + \frac{\alpha}{c})}. \quad (3.12)$$

Now we see from (3.9) and (3.11) that neither q_j nor x_j transform as the coordinates of the Minkowski space under a space translation –in fact they are changed by amounts depending on the particle positions and momenta. Further, although the rapidities have the correct transformation (3.12) that of the Minkowski positions is very different from the ordinary Lorentz boost. We conclude therefore that the theory is not relativistically invariant in the naive sense suggested by the “non-relativistic” limit given by Ruijsenaars and Schneider. Of course the details of the above verification for the non-invariance under the inhomogeneous Lorentz transformation have depended on our identification of the Minkowski coordinates and momenta, but without giving these explicitly the Ruijsenaars-Schneider theory cannot be said to describe relativistic dynamics.

We have argued that the Hamiltonian dynamics of the Ruijsenaars-Schneider theory is not invariant under Einstein’s special theory of relativity in the naive sense suggested by the “non-relativistic” limit given by Ruijsenaars and Schneider. As such we believe the description of these models as “relativistic Calogero-Moser” systems is thoroughly misleading. Indeed, the “non-relativistic” limit of these models requires an explicit scaling of the dimensionful coupling constant A above, and it is unclear why this should be described as a “non-relativistic” limit at all. It seems far more sensible to view the models as one-parameter generalisations of the Calogero-Moser systems.

3.1 Many “times” and Poincaré Invariance

It remains to explain in what sense Ruijsenaars-Schneider theory evidences Poincaré invariance. For such an invariance we require several “times” and their corresponding flows H_k . These times will be our coordinates. Now the dynamical variables evolve according to (1.1)

$$(q(t_1, t_2, \dots, t_l), \theta(t_1, t_2, \dots, t_l)) = \exp \left(\sum_{j=1}^l t_j H_j \right) (q(0), \theta(0)), \quad (3.13)$$

and because we have several times we are not really dealing with dynamics. Thus using our description (2.20,2.21,2.22) we see the solitons of the KdV equation evolve according to H_1 and H_3 and we have $q_j = q_j(t_1, t_3)$. Similarly the solitons of the SG equation evolve according to H_1 and H_{-1} and we have $q_j = q_j(t_{-1}, t_1)$. In the further restricted setting when we are dealing with flows H_k and H_{-k} it is possible to consider the associated Poincaré algebra (2.18). This is what distinguishes between the various soliton equations: although we may associate the Ruijsenaars-Schneider Hamiltonian (2.2) with solitons of each of the KdV, mKdV and SG equations for example, only the SG equation has a second flow that yields an associated Poincaré algebra. It remains to check that the “boost” does indeed behave correctly. Of course we always have that

$$e^{\alpha\mathcal{B}} e^{t_k \mathcal{H}_k - t_{-k} \mathcal{P}_k} e^{-\alpha\mathcal{B}} = e^{t'_k \mathcal{H}_k - t'_{-k} \mathcal{P}_k},$$

where

$$\begin{pmatrix} t'_k \\ t'_{-k} \end{pmatrix} = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix} \begin{pmatrix} t_k \\ t_{-k} \end{pmatrix},$$

but when we further have that $e^{\alpha\mathcal{B}} q_j(0) = q_j(0)$ (i.e. when q_j behaves as a Lorentz scalar) we see that

$$e^{\alpha\mathcal{B}} q_j(t_k, t_{-k}) = q_j(t'_k, t'_{-k}) \quad (3.14)$$

and we have an action of the Poincaré algebra on our coordinates. Using (3.12) we see for example that this is true for the $H_{\pm 1}$ flows for the SG equation. It is in this sense that the Ruijsenaars-Schneider theory is said to evidence Poincaré invariance, but this is very different from relativistic particle dynamics.

Let us further consider the SG example where $(t_1, t_{-1}) = (t, x)$. Here we have

$$[q(t, x), \theta(t, x)]_j = [\exp(tH - xP)(q(0), \theta(0))]_j, \quad j = 1, \dots, N, \quad (3.15)$$

in which H and P are given by (2.2) and (2.3), respectively. In this very specific setting, because the q_j behave as Lorentz scalar scalars, we may define a “trajectory” via

$$q_j(t, x_j(t)) = 0.$$

Ruijsenaars and Schneider show that this specifies $x_j(t)$ for all time and (3.14) shows these trajectories are Lorentz invariant. (Indeed we could have set q_j to equal any constant with a similar result; the choice $q_j = 0$ is motivated by the fact that asymptotically these correspond to the peaks of the solitons.) Now although these “trajectories” are relativistically invariant we again emphasise that they have not been presented as relativistically invariant *dynamics*.

Before closing this section a further comment is in order. We have seen that the two “times”, t_1 and t_{-1} , or t and x , are necessary for the Poincaré invariance of the Ruijsenaars-Schneider model. So, logically both of these times should be carried over to its non-relativistic limit, the Calogero-Moser models. Under the x evolution we simply have

$$x_j(t, x) = x_j(t) + x, \quad p_j(t, x) = p_j(t).$$

In the physical interpretation of the Calogero-Moser models the presence of this additional “time” x is both redundant and rather confusing. This is another reason why we believe that the description of the Ruijsenaars-Schneider model as “relativistic Calogero-Moser models” is misleading.

4 Ruijsenaars-Schneider Theory and Field Theory

In the previous section we have seen that when one considers the $H_{\pm 1}$ (or equivalently the H and P) flows associated with for example the SG equation, the $q_j(t, x)$ given by (3.15) behave as Lorentz scalars. One might naively be tempted to think these describe an N -component scalar field in the $1 + 1$ dimensional Minkowski space (t, x) , and similarly that $\theta(t, x)$ are *dynamical* fields of some $1 + 1$ dimensional theory. We will now argue that this is not really the case and discuss the physical content of the solutions $q(t, x)$, $\theta(t, x)$ of the Ruijsenaars-Schneider model.

An ordinary field variable, say $\phi(t, x)$ describes a dynamical system with infinitely many degrees of freedom (one associated to each point x of space). At equal times these degrees of freedom are independent of each other and this is expressed by the Poisson bracket (or commutation) relation

$$\{\phi(t, x), \phi(t, y)\} = 0, \quad ([\phi(t, x), \phi(t, y)] = 0).$$

In other words, in an initial value problem ($t = 0$), the initial values $\phi(0, x)$ can be chosen arbitrarily. On the other hand, as is clear from (3.15) $q(0, x)$ and $\theta(0, x)$ are severely constrained. They are the solutions of

$$\begin{aligned} \frac{\partial}{\partial x} q_j(0, x) &= \{q_j(0, x), P\}, \\ \frac{\partial}{\partial x} \theta_j(0, x) &= \{\theta_j(0, x), P\}, \quad t = 0, \quad -\infty < x < \infty, \end{aligned} \tag{4.1}$$

with the condition $q_j(0, 0) = q_j(0)$ and $\theta_j(0, 0) = \theta_j(0)$. It is obvious that such constraints can never be imposed on a relativistic field variable $\phi(0, x)$ without breaking causality.

Further, the “time-evolution” of $q_j(t, x)$ and $\theta_j(t, x)$ are also very different from those of a relativistic field. At time t , $q_j(t, x)$ and $\theta_j(t, x)$ are solely determined by the “initial data” $\{q_k(0, x), \theta_k(0, x)\}$, $k = 1, \dots, N$ depending only on the *same* x , since they are solutions of

$$\begin{aligned}\frac{\partial}{\partial t} q_j(t, x) &= \{q_j(t, x), H\}, \\ \frac{\partial}{\partial t} \theta_j(t, x) &= \{\theta_j(t, x), H\},\end{aligned}\tag{4.2}$$

with the initial value $q_j(0, x)$ and $\theta_j(0, x)$. This is in marked contrast with a dynamical relativistic field, in which $\phi(t, x)$ depends on the initial data $\phi(0, y)$ within the past light-cone, ie, $x - ct \leq y \leq x + ct$.

Indeed, given the $2N$ initial conditions $q_j(0, 0) = q_j(0)$ and $\theta_j(0, 0) = \theta_j(0)$ at any one point, the solutions $q_j(t, x)$, $j = 1, \dots, N$ of Ruijsenaars-Schneider models are then specified “globally”. The properties we have just described show that the the solutions $q(t, x)$, $\theta(t, x)$ of the Ruijsenaars-Schneider model are not describing the dynamical time-evolution typical of field theory. Indeed this lack of “dynamics” bears many of the hallmarks of a topological field theory: although we cannot as yet make this precise we conclude the section with a Lax pair encoding the evolution with respect to the various flows. *Let V be an $N \times N$ diagonalisable matrix such that*

$$\partial_{\pm} V = \Lambda_{\pm} V + V \Lambda_{\pm} \quad [\Lambda_+, \Lambda_-] = 0,\tag{4.3}$$

and Λ_{\pm} are constant. Then with $Q = UVU^{-1} = \text{diag}(\exp(q_1), \dots, \exp(q_N))$, $M_{\pm} = \partial_{\pm} U U^{-1}$ and $L_{\pm} = U \Lambda_{\pm} U^{-1}$ we have

$$[D_+, D_-] = 0\tag{4.4}$$

where

$$D_{\pm} = \partial_{\pm} + \begin{pmatrix} -M_{\pm} - L_{\pm} & \mu^{\pm 1} Q \\ 0 & L_{\pm} - M_{\pm} \end{pmatrix},\tag{4.5}$$

and μ is a spectral parameter.

5 Uncertainty Principle

Let us now examine the possibility of ‘reducing’ an integrable relativistic quantum field theory with factorisable S -matrices to a collection of fixed particle number quantum mechanical systems; this was mentioned as means of motivation at the outset of the work of Ruijsenaars and Schneider. The known exact factorisable S -matrices of, for example, sine-Gordon theory [26] and affine Toda field theory [27, 28] have been obtained as solutions of the Yang-Baxter equation and/or bootstrap equation satisfying analyticity, unitarity and crossing symmetry.

Now in a crossing symmetric quantum field theory a field operator ϕ_j annihilates particles of species j and creates their anti-particles. Therefore any interaction term in the Lagrangian of a crossing symmetric field theory changes the particle numbers. This is in sharp contrast with non-relativistic quantum field theories (for example, the non-linear Schrödinger theory) in which the interaction term $(\bar{\psi}\psi)^2$ is manifestly particle number preserving: ψ annihilates a particle and $\bar{\psi}$ creates a particle.

In contrast to the no-particle production which is the hallmark of an integrable classical field theory and based on its infinite number of conservation laws, in relativistic quantum field theory this property is only guaranteed between the two asymptotic states at $t = \infty$ and $t = -\infty$ [29]. In other words, the results of measuring any classically conserved quantity over a finite time interval will fluctuate because of the uncertainty principle of the quantum theory. In particular, the particle numbers will not be constant over time due to the various virtual processes caused by the above mentioned particle number non-preserving interactions. Various field theoretical calculations of the S-matrices and other quantities [30] in affine Toda field theory show this fact explicitly. Thus we arrive at the conclusion that a ‘reduction’ of a solvable relativistic quantum field theory to a collection of fixed particle number (relativistic) quantum mechanical systems is impossible.

6 Summary and Discussion

We have discussed various aspects of the Ruijsenaars-Schneider model. In particular we have argued that these models are most naturally viewed as a one-parameter generalisation of the Calogero-Moser models which should not be described as a “relativistic” generalisation: the model is not in fact “relativistically invariant” in the sense dictated by the “non-relativistic” limit. Further we have compared the many (compatible) “times” formulation –in which certain models are Poincaré invariant– with standard field theory. In this context the Ruijsenaars-Schneider model does not describe a dynamical field theory. This is entirely natural in the soliton setting that gives rise to the model, for the Ruijsenaars-Schneider equations simply describe a *single* solution to the associated soliton-bearing PDE in an analogous manner to the inverse scattering transform. We have also discussed constraints that the uncertainty principle places on any possible linkage between integrable quantum field theories with exact factorisable S-matrices and integrable particle dynamics. In spite of the difficulties related to the “relativistic” interpretation, we again emphasise the importance of these models, an importance we believe stems from the natural matrix equations associated

with the models. The Ruijsenaars-Schneider equations in this setting are not only generic but useful. The work on Ruijsenaars-Schneider models with spin degrees of freedom is still in its infancy and we would like the connections between such models and the affine Toda field theories to be pursued both algebraically and physically.

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References

- [1] F. Calogero, *J. Math. Phys.*, **10** (1969) 2191, 2197, **12** (1971) 419; *Lett. Nuovo Cimento* (2) **13** (1975) 411, **16** (1976) 77; B. Sutherland, *Phys. Rev.*, **A5** (1972) 1372; J. Moser, *Adv. Math.* **16** (1975) 1.
- [2] S. Iso and S. J. Rey, *Phys. Lett.* **B352** (1995) 111; H. Awata, Y. Matsuo and T. Yamamoto, *J. Phys.* **A29** (1996) 3089.
- [3] A. Gorsky and N. Nekrasov, *Nucl. Phys.* **B414** (1994) 213, *Elliptic Calogero-Moser system from two dimensional current algebra* [hep-th/9401021](#).
- [4] R. Camassa and D.D. Holm, *Phys. Rev. Lett.* **71** (1993) 1661; L. Brink, T.H. Hansson, S. Konstein and M.A. Vasiliev, *Nucl. Phys.* **B401** (1993) 591; I. Andrić, V. Bardek and L. Jonke, *Phys. Lett.* **B357** (1995) 374.
- [5] M.A. Olshanetsky and A.M. Perelomov, *Phys. Reps.* **71** (1981) 313; P. Pechukas, *Phys. Rev. Lett.* **51** (1983) 943; K. Nakamura and M. Lakshmanan, *Phys. Rev. Lett.* **57** (1987) 1661.
- [6] D.V. Choodnovsky and G.V. Choodnovsky, *Nuovo Cimento* **40B** (1977) 339; H. Airault, H. McKean and J. Moser, *Comm. Pure Appl. Math.* **30** (1977) 95; I.M. Krichever, *Funct. anal. i pril.* **12** (1978) 76; T. Shiota, *J. Math. Phys.* **35** (1995) 5844.

- [7] M. A. Olshanetsky and A. M. Perelomov, *Phys. Reps.* **94** (1983) 313.
- [8] M. Caselle *Phys. Rev. Lett.* **74** (1995) 2776.
- [9] E. Bergshoeff and M. Vasiliev, *Int. J. Mod. Phys.* **A10** (1995) 3477; V. Marotta and A. Sciarrino, *Nucl. Phys.* **B476** (1996) 351; R. Caracciolo, M. Frau, A. Lerda, S. Sciuto and G. R. Zemba, *The extended conformal theory of the Calogero-Sutherland model*, To appear in the proceedings of the IV Meeting ‘Common Trends in Condensed Matter and High Energy Physics’, Chia Laguna, Cagliari, Italy, 3-10 Sep. 1995, [hep-th/9604009](#).
- [10] A. Matsuo, *Invent. Math.* **110** (1992) 95; D. Bernard, K. Hikami, and M. Wadati, *Nankai Math. Physics* (1994) 1; G. Felder and A. P. Veselov, *Commun. Math. Phys.* **160** (1994) 259; H. Awata, Y. Matsuo, S. Odake and J. Shiraishi, *Nucl. Phys.* **B449** (1995) 347; M. Adler, T. Shiota and P. van Moerbeke, *Phys. Lett.* **A208** (1995) 67; C.-H. Ahn and S.-K. Nam, *Phys. Lett.* **B378** (1996) 107; T. H. Baker and P. J. Forrester, *The Calogero-Sutherland Model and Generalized Classical Polynomials*, [solv-int/9608004](#), *The Calogero-Sutherland Model and Polynomials with Prescribed Symmetry*, [solv-int/9609010](#).
- [11] S. N. M. Ruijsenaars, *Ann. of Phys.* **126** (1980) 399.
- [12] S. N. M. Ruijsenaars and H. Schneider, *Ann. of Phys.* **170** (1986) 370.
- [13] A. Gorsky and N. Nekrasov, *Nucl. Phys.* **B436** (1995) 582.
- [14] Y. B. Suris, *Phys. Lett.* **A225** (1997) 253; J. Avan and G. Rollet, *The Classical r -Matrix for the Relativistic Ruijsenaars-Schneider System*, [hep-th/9510166](#); F. W. Nijhoff, V. B. Kuznetsov, E. K. Sklyanin and O. Ragnisco, *J. Phys.* **A29** (1996) L333.
- [15] I. Krichever and A. Zabrodin, *Spin generalization of the Ruijsenaars-Schneider model. non-abelian 2D Toda chain and representations of Sklyanin algebra*, [hep-th/9505039](#); H. Konno, *J. Phys.* **A29** (1996) L191.
- [16] J. Gibbons and T. Hermesen, *Physica* **11D** (1983) 337. E. Billey, J. Avan and O. Babelon, *Phys. Lett.* **A188** (1994) 263; I. Krichever, O. Babelon, E. Billey and M. Talon, *Spin generalization of the Calogero-Moser system and the Matrix KP equation*, [hep-th/9411160](#).
- [17] H. W. Braden and A. N. W. Hone, *Phys. Lett.* **B380** (1996) 296.

- [18] O. Babelon and D. Bernard, *Phys. Lett.* **B317**, (1993) 363; O. Babelon, D. Bernard and F. A. Smirnov, *Commun. Math. Phys.* **182** (1996) 319.
- [19] S. N. M. Ruijsenaars, *Commun. Math. Phys.* **110** (1987) 191.
- [20] S. N. M. Ruijsenaars, *Commun. Math. Phys.* **115** (1988) 127.
- [21] P. A. M. Dirac, *Rev. Mod. Phys.* **21** (1949) 392.
- [22] N. Kemmer, *Helv. Phys. Acta* **10** (1937) 48.
- [23] W. Glöckle, Y. Nogami and I. Fukui, *Phys. Rev.* **D35** (1987) 584; Y. Munakata, in Proceedings of International Symposium on “Extended Objects and Bound States”, Karuizawa, Japan, 19 ~ 21 March 1992.
- [24] E. C. G. Sudarshan and N. Makunda, *Classical Dynamics: A Modern Perspective*, John Wiley, N.Y. 1974.
- [25] F. W. Nijhoff, O. Ragnisco and V. B. Kuznetsov, *J. Phys.* **A29** (1996) L333, *Commun. Math. Phys.* **176** (1996) 681; O. Ragnisco and Yu. B. Suris, *Integrable discretizations of the spin Ruijsenaars-Schneider models*, solv-int/9605001.
- [26] A. B. Zamolodchikov and Al. B. Zamolodchikov, *Ann. Phys.* **120** (1979) 253.
- [27] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, *Phys. Lett.* **B227** (1989) 411, *Nucl. Phys.* **B338**, (1990) 689; E. Corrigan, P. E. Dorey and R. Sasaki, *Nucl. Phys.* **B408** (1993) 579.
- [28] P. Christe and G. Mussardo, *Nucl. Phys.* **B330**, (1990) 465, *Int. J. Mod. Phys.* **A5**, (1990) 4581; G. W. Delius, M. T. Grisaru and D. Zanon, *Nucl. Phys.* **B382** (1992) 365.
- [29] See for example, V. B. Berestetskii, E. M. Lifshitz and L. P. Pitaevskii, *Quantum Electrodynamics*, Landau and Lifshitz Course Vol. 4 §1, Pergamon Press (1982).
- [30] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, *Nucl. Phys.* **B356**, (1991) 469; H. W. Braden and R. Sasaki, *Phys. Lett.* **B255** (1991) 343, *Nucl. Phys.* **B379** (1992) 377.